# Summary of Relationships for Two-Dimensional Rigid-Body Dynamics

All of the following results are given without proof:

### Angular Momentum

The angular momentum $\mathbf{L}_A$ of a system of $n$ particles about a point $A$ is defined as

$$\mathbf{L}_A \equiv \sum_{i=1}^{i=n} m_i \, \mathbf{r}_i \times \mathbf{v}_i, \qquad (1)$$

where $\mathbf{r}_i$ and $\mathbf{v}_i$ are the position and velocity, respectively, of particle $i$ with respect to $A$. In Eq. (1), point $A$ as well as the particles can be moving in any manner whatsoever. For a rigid body, where the inter-particle distances are fixed, Eq. (1) becomes

$$\mathbf{L}_A = I_{cm}\boldsymbol{\omega} + m\, \mathbf{r}_{cm} \times \mathbf{v}_{cm}, \qquad (2)$$

where $I_{cm}$ is the moment of inertia about an axis through the c.m. perpendicular to the plane in which the object is rotating, $\mathbf{r}_{cm}$ is the vector from $A$ to the c.m., $m$ is the total mass, and $\mathbf{v}_{cm}$ is the velocity of the c.m. relative to point $A$. Eq. (2) can be used directly or reduces usefully in three commonly occurring cases:

(i) If $A$ is fixed, then, again,

$$\mathbf{L}_A = I_{cm}\boldsymbol{\omega} + m\, \mathbf{r}_{cm} \times \mathbf{v}_{cm}, \qquad (2)$$

but now, $\mathbf{v}_{cm}$ is simply the velocity of the c.m. in the frame of reference in which the problem is posed.

(ii) If $A$ is a fixed point and the motion of the rigid body is purely rotational about $A$, then,

$$\mathbf{L}_A = I_A\boldsymbol{\omega}\ . \qquad (3)$$

(iii) If $A$ is the c.m., then the second term on the right in eq. (2) vanishes because $\mathbf{r}_{cm}$ = the displacement vector from $A$ to the c.m. and is zero. Thus,

$$\mathbf{L}_{cm} = I_{cm}\boldsymbol{\omega}\ . \qquad (4)$$

Note that Eq. (4) holds even if the c.m. is either moving or accelerating. However, if the c.m. is fixed, $I_{cm}\boldsymbol{\omega}$ is the angular momentum about *any* fixed point, as can be easily seen from the fact that $\mathbf{v}_{cm} = 0$ in Eq. (2).

*(continued on other side)*

## Torque

If point $A$ is the center of mass of the system, a fixed point in space, or the instantaneous center of rotation of a rigid body, then

$$\boldsymbol{\tau}_A = d/dt\,(\mathbf{L}_A). \tag{5}$$

There are two commonly occurring cases for which Eq. (5) reduces usefully:

(i) Conservation of angular momentum: If $A$ is a point about which there is no torque, then $\mathbf{L}_A$ is conserved, and

$$\mathbf{L}_A = \mathbf{L}'_A \tag{6}$$

(ii) If there are external torques about point $A$, then

$$\boldsymbol{\tau}_A = I_A\boldsymbol{\alpha}. \tag{7}$$

*Remember, for (i) and (ii) above to hold,* A *must be a fixed point, the c.m., or the instantaneous center of rotation of a rigid body.*

## Kinetic Energy

The KE of a system of particles is KE $\equiv 1/2 \sum m_i v_i^2$. For a rigid body this becomes

$$KE = 1/2\, m_{tot}\, v_{cm}^2 + 1/2\, I_{cm}\omega^2.$$

That is, the KE can be thought of as having two contributions: (1) KE of the translational energy of the total mass of the body moving at a velocity of its c.m. and (2) KE of its rotational energy about the center of mass. If the rigid body is solely rotating about a fixed point $A$, then,

$$KE = 1/2\, I_A\omega^2.$$

## Prescription for Solving Rigid-Body-Dynamics Problems

If there are no external forces on a system, then both angular momentum and linear momentum are conserved. Energy is not conserved if there is internal friction or if inelastic collisions occur. If no work is done by non-conservative forces (internal or external), then energy is conserved. If there is a force on the system, even if that force does no work, it will generally produce a torque. Then it is best to use $\boldsymbol{\tau} = I\boldsymbol{\alpha}$ and $\mathbf{F} = m\,\mathbf{a}_{cm}$. Often, judicious choice of the point about which to take torques saves time. When using $\boldsymbol{\tau} = I\boldsymbol{\alpha}$, only use either a fixed point of the object (if one exists), the c.m., or the instantaneous center of rotation (the point on the surface instantaneously in contact with the rolling object). When the total external torque about one of these three points is zero, angular momentum about that point is conserved. When an object of circular cross-section rolls w/o slipping, $v_{cm} = R\omega$, and $a_{cm} = R\alpha$. Moreover, in this case, friction does no work. Whereas torque, angular momentum, and angular velocity are defined in terms of three-dimensional vectors, in a two-dimensional problem, these quantities are most easily described as scalars, with counter-clockwise = positive and clockwise = negative. This description is possible because, for a two-dimensional problem, each of these three vector quantities will be in the $+z$ direction for a CCW rotational sense and in the $-z$ direction for a CW rotational sense.